$$\begin{aligned}\phi &= F(G \star \phi) = F(\overline{\phi}) \\ &= \overline{\phi} + \theta \frac{\partial \overline{\phi}}{\partial t} + \overline{\Delta}_l \frac{\partial \overline{\phi}}{\partial x_l} + \overline{\Delta}_{lm} \frac{\partial^2 \overline{\phi}}{\partial x_{lm}} + \dots \quad , \end{aligned} \tag{2.29}$$

where θ and $\overline{\Delta}_l$ are some time and space scales, respectively. Differential filters can be grouped into several classes: elliptic, parabolic or hyperbolic filters. In the framework of a generalized space-time filtering, Germano [111, 112, 114] recommends using a parabolic or hyperbolic time filter and an elliptic space filter, for reasons of physical consistency with the nature of the Navier–Stokes equations. It is recalled that a filter is said to be elliptic (resp. parabolic or hyperbolic) if F is an elliptic (resp. parabolic, hyperbolic) operator. Examples are given below [117].

Time Low-Pass Filter. A first example is the time low-pass filter. The associated inverse differential relation is :

$$\phi = \overline{\phi} + \theta \frac{\partial \overline{\phi}}{\partial t} \quad . \tag{2.30}$$

The corresponding convolution filter is:

$$\overline{\phi} = \frac{1}{\theta} \int_{-\infty}^{t} \phi(x, t') \exp\left(-\frac{t - t'}{\theta}\right) dt' \quad . \tag{2.31}$$

It is easily seen that this filter commutes with time and space derivatives. This filter is causal, because it incorporates no future information, and therefore is applicable to real-time or post-processing of the data.

Elliptic Filter. An elliptic filter is obtained by taking:

$$\phi = \overline{\phi} - \overline{\Delta}^2 \frac{\partial^2 \overline{\phi}}{\partial x_l^2} \quad . \tag{2.32}$$

It corresponds to a second-order elliptic operator, which depends only on space. The convolutional integral form is:

$$\overline{\phi} = \frac{1}{4\pi \overline{\Delta}^2} \int \frac{\phi(\xi, t)}{|x - \xi|} \exp\left(-\frac{|x - \xi|}{\overline{\Delta}}\right) d\xi \quad . \tag{2.33}$$

This filter satisfies the three previously mentionned basic properties.

Parabolic Filter. A parabolic filter is obtained taking

$$\phi = \overline{\phi} + \theta \frac{\partial \overline{\phi}}{\partial t} - \overline{\Delta}^2 \frac{\partial^2 \overline{\phi}}{\partial x_l^2} \quad , \tag{2.34}$$

yielding

$$\overline{\phi} = \frac{\sqrt{\theta}}{(4\pi)^{3/2} \overline{\Delta}^3} \int_{-\infty}^{t} \int \frac{\phi(\xi, t)}{(t - t')^{3/2}} \exp\left(-\frac{(x - \xi)^2 \theta}{4 \overline{\Delta}^2 (t - t')} - \frac{t - t'}{\theta}\right) d\xi dt' \quad . \tag{2.35}$$

It is easily verified that the parabolic filter satistifies the three required properties.

Convective and Lagrangian Filters. A convective filter is obtained by adding a convective part to the parabolic filter, leading to:

$$\phi = \overline{\phi} + \theta \frac{\partial \overline{\phi}}{\partial t} + \theta V_l \frac{\partial \overline{\phi}}{\partial x_l} - \overline{\Delta}^2 \frac{\partial^2 \overline{\phi}}{\partial x_l^2} \quad , \tag{2.36}$$

where $\mathbf{V}$ is an arbitrary velocity field. This filter is linear and constant preserving, but commutes with derivatives if and only if $\mathbf{V}$ is uniform. A Lagrangian filter is obtained when $\mathbf{V}$ is taken equal to $\mathbf{u}$, the velocity field. In this last case, the commutation property is obviously lost.

2.1.5 Three Classical Filters for Large-Eddy Simulation

Three convolution filters are ordinarily used for performing the spatial scale separation. For a cutoff length $\overline{\Delta}$, in the mono-dimensional case, these are written:

– Box or top-hat filter:

$$G(x - \xi) = \begin{cases} \dfrac{1}{\overline{\Delta}} & \text{if } |x - \xi| \le \dfrac{\overline{\Delta}}{2} \\ 0 & \text{otherwise} \end{cases} \quad , \tag{2.37}$$

$$\widehat{G}(k) = \frac{\sin(k\overline{\Delta}/2)}{k\overline{\Delta}/2} \quad . \tag{2.38}$$

The convolution kernel G and the transfer function $\widehat{G}$ are represented in Figs. 2.1 and 2.2, respectively.

– Gaussian filter:

$$G(x - \xi) = \left(\frac{\gamma}{\pi \overline{\Delta}^2}\right)^{1/2} \exp\left(\frac{-\gamma |x - \xi|^2}{\overline{\Delta}^2}\right) \quad , \tag{2.39}$$

$$\widehat{G}(k) = \exp\left(\frac{-\overline{\Delta}^2 k^2}{4\gamma}\right) \quad , \tag{2.40}$$

in which γ is a constant generally taken to be equal to 6. The convolution kernel G and the transfer function $\widehat{G}$ are represented in Figs. 2.3 and 2.4, respectively.

– Spectral or sharp cutoff filter:

$$G(x - \xi) = \frac{\sin\left(k_c (x - \xi)\right)}{k_c (x - \xi)}, \text{ avec } k_c = \frac{\pi}{\overline{\Delta}} \quad , \tag{2.41}$$

$$\widehat{G}(k) = \begin{cases} 1 & \text{if } |k| \le k_c \\ 0 & \text{otherwise} \end{cases} \quad . \tag{2.42}$$

The convolution kernel G and the transfer function $\widehat{G}$ are represented in Figs. 2.5 and 2.6, respectively.

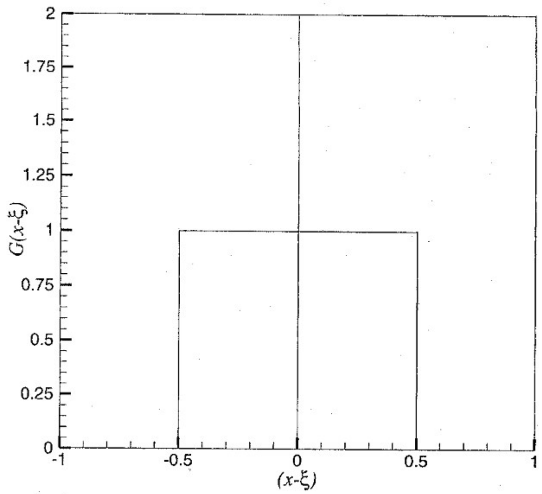

Fig. 2.1. Top-hat filter. Convolution kernel in the physical space normalized by $\overline{\Delta}$.

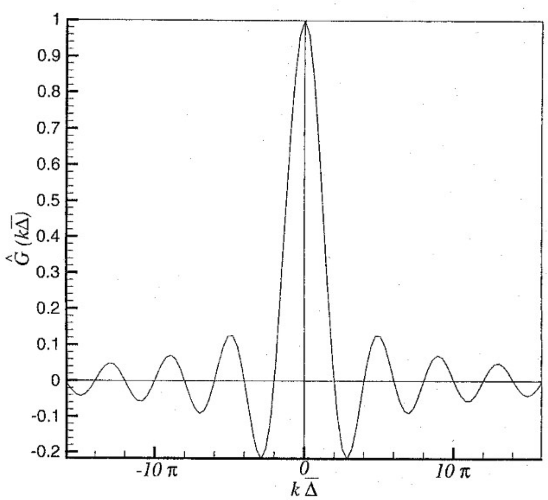

Fig. 2.2. Top-hat filter. Associated transfer function.

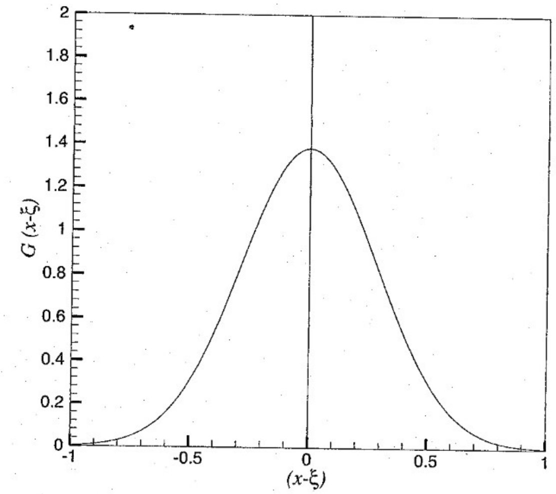

Fig. 2.3. Gaussian filter. Convolution kernel in the physical space normalized by $\overline{\Delta}$.

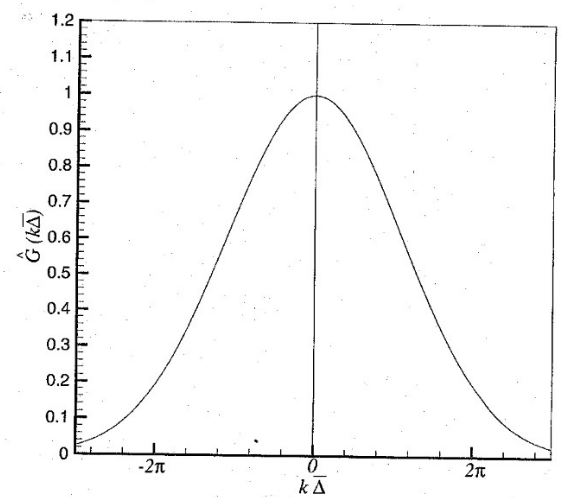

Fig. 2.4. Gaussian filter. Associated transfer function.

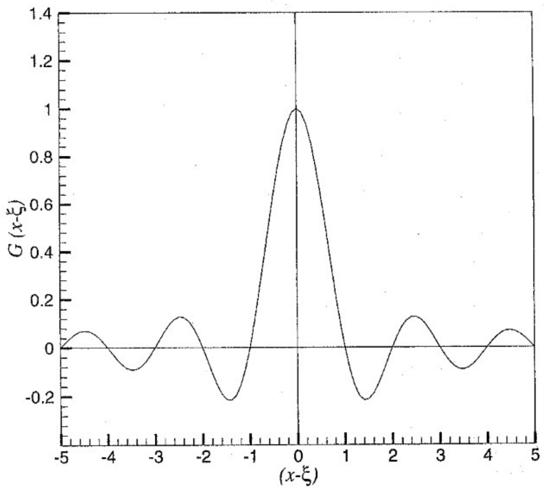

Fig. 2.5. Sharp cutoff filter. Convolution kernel in the physical space.

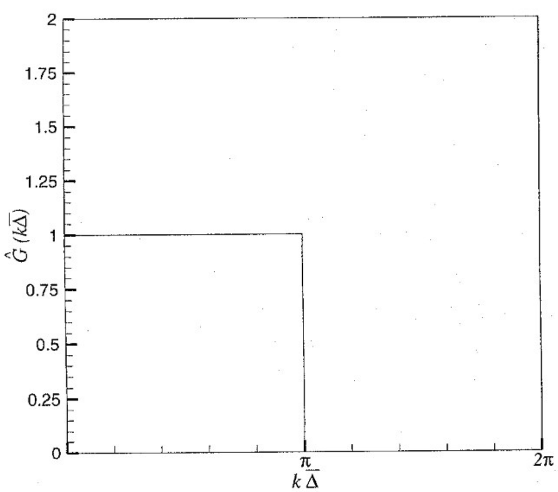

Fig. 2.6. Sharp cutoff filter. Associated transfer function.

It is trivially verified that the first two filters are positive while the sharp cutoff filter is not. The top-hat filter is local in the physical space (its support is compact) and non-local in the Fourier space, inversely from the sharp cutoff filter, which is local in the spectral space and non-local in the physical space. As for the Gaussian filter, it is non-local both in the spectral and physical spaces. Of all the filters presented, only the sharp cutoff has the property:

$$\underbrace{\widehat{G} \cdot \widehat{G} \ldots \cdot \widehat{G}}_{n \text{ times}} = \widehat{G}^n = \widehat{G} \quad ,$$

and is therefore idempotent in the spectral space. Lastly, the top-hat and Gaussian filters are said to be smooth because there is a frequency overlap between the quantities $\overline{\mathbf{u}}$ and $\mathbf{u}'$.

2.2 Extension to the Inhomogeneous Case

2.2.1 General

In the above explanations, it was assumed that the filter is homogeneous and isotropic. These assumptions are at time too restrictive for the resulting conclusions to be usable. For example, the definition of bounded fluid domains forbids the use of filters that are non-local in space, since these would no longer be defined. The problem then arises of defining filters near the domain boundaries. At the same time, there may be some advantage in varying the filter cutoff length to adapt the structure of the solution better and thereby ensure optimum gain in terms of reducing the number of degrees of freedom in the system to be resolved.

From relation (2.1), we get the following general form of the commutation error for a convolution filter $G(y, \overline{\Delta}(x,t))$ on a domain Ω [105, 121]:

$$\left[\frac{\partial}{\partial x}, G\star\right]\phi = \frac{\partial}{\partial x}(G \star \phi) - G \star \frac{\partial \phi}{\partial x} \; . \tag{2.43}$$

The first term of the right hand side of (2.43) can be expanded as

$$\frac{\partial}{\partial x}(G \star \phi) = \frac{\partial}{\partial x}\int_{\Omega} G(x-\xi, \overline{\Delta}(x,t))\phi(\xi,t)d\xi \tag{2.44}$$

$$= \left(\frac{\partial G}{\partial \overline{\Delta}} \star \phi\right)\frac{\partial \overline{\Delta}}{\partial x} + \int_{\partial\Omega} G(x-\xi, \overline{\Delta}(x,t))\phi(\xi,t)n(\xi)ds$$

$$+ G \star \frac{\partial \phi}{\partial x} \quad , \tag{2.45}$$

where $n(\xi)$ is the outward unit normal vector to the boundary of Ω, $\partial\Omega$, yielding